LARGE CARDINALS IN GENERAL TOPOLOGY I

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Large cardinals

The existence of a large cardinal κ should not be inconsistent with ZFC.

If ZFC is consistent, then ZFC + "the large cardinal κ does not exist" is consistent.

Measurable cardinals
S.Banach, K.Kuratowski, S.Ulam (Lvov 1929-1930)

Stanislaw Marcin Ulam, 1909-1984

Stefan Banach 1892–1945

Kazimierz Kuratowski 1896–1980

Definition

For an infinite cardinal κ we say that a measure μ on A is κ-additive if $\mu(\cup_\lambda A_\alpha) = \sum_\lambda \mu(A_\alpha)$ whenever $\{A_\alpha\}_\lambda$ is a disjoint collection of subsets of A and $\lambda < \kappa$.

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All our measures are defined on all subsets of some set. We shall assume that measures are non-trivial in the sense that the measure of the whole set is not zero, while measures of points are zero.

Measures with ranges equal to $\{0,1\}$ are called two-valued.

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If $\kappa = \omega$ (or $\kappa = \omega_1$), we speak about finitely additive (or countably additive) measure.

Definition (Measurable cardinals)

A cardinal number κ is said to be real-measurable if there is a κ-additive measure on the set κ.
A cardinal number κ is said to be measurable if there is a κ-additive two-valued measure on the set κ.

The class of measurable cardinals will be ordered: $\omega = \mathfrak{m}_0 < \mathfrak{m}_1 < \dots$.
The first uncountable real-measurable cardinal is denoted as $\mathfrak{m}_\mathbb{R}$.

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The first uncountable real-measurable cardinal is denoted as $\mathfrak{m}_{\mathbb{R}}$.

Theorem

Every real measurable cardinal is inaccessible.
Every measurable cardinal is strongly inaccessible.
Any real measurable cardinal is measurable provided it is bigger than 2^ω.

Theorem (R.N.Solovay)

The consistencies of $\{\mathrm{ZFC} + \exists \mathfrak{m}_1\}, \{\mathrm{ZFC} + \exists \mathfrak{m}_\mathbb{R}\}, \{\mathrm{ZFC} + (\exists \mathfrak{m}_\mathbb{R} \leq 2^\omega)\}$ *are equivalent.*

For a cardinal κ, we denote by $\mathfrak{m}(\kappa)$ the first measurable cardinal bigger than κ or a symbol ∞ bigger than any cardinal if there is no measurable cardinal bigger than κ.

Theorem

Every κ*-additive measure is* $\mathfrak{m}(\kappa)$*-additive.*

Corollary

The first uncountable measurable cardinal is the first uncountable cardinal admitting a countably additive measure.

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Ultrafilters

Definition (κ-completeness of filters)

For an infinite cardinal κ, a filter $\mathcal{F}$ of subsets of A is said to be κ-complete if $\bigcap_\lambda A_\alpha \in \mathcal{F}$ whenever $\{A_\alpha\}_\lambda \subset \mathcal{F}$ and $\lambda < \kappa$.

Instead of ω_1-complete filters we speak about countably complete filters.

If μ is a two-valued κ-additive measure on a set A then $\{P \subset A; \mu(P) = 1\}$ is a free κ-complete ultrafilter on A.

Conversely, if $\mathcal{F}$ is a free κ-complete ultrafilter then μ with value 1 at sets from $\mathcal{F}$ and zero otherwise is a two-valued κ-additive measure on A.

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A cardinal κ is measurable iff there exists a free κ-complete ultrafilter on the set κ.

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Measurable cardinals and topology

In this part, we shall work in Hausdorff completely regular (i.e., Tikhonov) spaces only.

Measurable cardinals and topology

Trivially: *A set* X *has cardinality less than* $\mathfrak{m}$ *iff every maximal filter on* $\exp X$ *is fixed provided it is* $\mathfrak{m}$*-complete.*

When X is a topological space, one can take some subclasses of $\exp X$ instead of $\exp X$, like the class of closed sets or of zero sets in X (i.e., sets of a form $f^{-1}(0), f : X \to \mathbb{R}$ continuous).

Theorem ($\mathfrak{m} = \omega$)

The following conditions for a topological space X are equivalent:

1. *X is compact.*
2. *Every ultrafilter $\mathcal{X}$ on X converges (i.e., $\bigcap_{\mathcal{X}} \overline{A} \neq \emptyset$).*
3. *For every ultrafilter $\mathcal{X}$ on X the filter $\{F \in \mathcal{X}; F$ is closed $\}$ in closed sets in X is fixed (has nonempty intersection).*
4. *Every maximal filter of closed sets in X is fixed (has nonempty intersection).*
5. *For every ultrafilter $\mathcal{X}$ on X the filter $\{F \in \mathcal{X}; F$ is a zero set $\}$ in zero sets in X is fixed (has nonempty intersection)*
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Zero sets

For every ultrafilter $\mathcal{X}$ on X the filter $\{F \in \mathcal{X}; F$ is a zero set $\}$ in zero sets in X is fixed (has nonempty intersection) provided it is κ-complete.

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Definition (κ-compact spaces, H.Herrlich)

A topological space X is said to be κ-compact if every maximal zero filter that is κ-complete, has nonempty intersection.

ω_1-compact space = realcompact space

Closed sets

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Definition (κ-ultracompact spaces, J.van der Slot)

A topological space X is said to be κ-ultracompact if every ultrafilter with κ-complete property for its closed sets converges.

Definition

A productive and closed-hereditary (i.e., epireflective) class $\mathcal{C}$ of spaces is said to be simple if there is $Z \in \mathcal{C}$ such that every $X \in \mathcal{C}$ can be embedded onto a closed subspace a power of Z. One says that $\mathcal{C}$ is generated by Z and the spaces from $\mathcal{C}$ are then called Z-compact spaces.

By $\mathcal{C}_\kappa$ we denote the class of κ-compact spaces.
By $\mathcal{U}_\kappa$ we denote the class of κ-ultracompact spaces.

The classes $\mathcal{C}_\omega, \mathcal{U}_\omega$ are simple, they coincide with the class of compact spaces and are generated by $[0, 1]$.

The class $\mathcal{C}_{\omega_1}$ is the simple class of realcompact spaces generated by $\mathbb{R}$.

1. Are the classes $\mathcal{C}_\kappa, \kappa \geq \omega_2$ simple?
2. Are the classes $\mathcal{U}_\kappa, \kappa \geq \omega_1$ simple?
3. What is a relation between $\mathcal{C}_\kappa$ and $\mathcal{U}_\kappa$?

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Theorem (MH)

The classes $\mathcal{C}_\kappa$ are simple. For any cardinal κ, the class $\mathcal{C}_{\kappa^+}$ is generated by $P_\kappa = [0,1]^\kappa \setminus \{1\}$. For limit κ the class $\mathcal{C}_\kappa$ is generated by $\prod_\kappa P_\lambda$.

Theorem (van der Slot, Z.Frolík)

The class of perfect images of spaces from $\mathcal{C}_\kappa$ coincides with the class $\mathcal{U}_\kappa$.

Theorem (MH)

If $\mathcal{C}$ is epireflective, closed under perfect images and contains a discrete space of cardinality μ then $\mathcal{C}$ is not a part of E-compact spaces for any space E of cardinality less than $\mathfrak{m}(\mu)$.

Corollary

1. *The classes $\mathcal{U}_\kappa, \omega < \kappa < \mathfrak{m}_1$, are not generated by a space of cardinality $< \mathfrak{m}_1$.*
2. *The classes $\mathcal{C}_\kappa, \omega < \kappa < \mathfrak{m}_1$, and $\mathcal{U}_\lambda$ are all different.*

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PROBLEMS

1. Is $\mathcal{C}_{\mathfrak{m}} = \mathcal{U}_{\mathfrak{m}}$ for measurable cardinals $\mathfrak{m}$?
2. Are the classes $\mathcal{U}_\kappa$ simple?

Similar situation

P.Nyikos: *The class of* $\mathbb{N}$*-compact spaces is a proper subclass of the class of all zerodimensional realcompact spaces. It does not contain the Prabir Roy's metric space* Δ *with* $\operatorname{ind}\Delta = 0, \operatorname{Ind}\Delta = 1$.

Problem: Is the class of all zerodimensional realcompact spaces simple?

A.Mysior

The class of all zerodimensional realcompact spaces is not generated by any space of Ulam non-measurable cardinality.

Problem

Is the class of all zerodimensional realcompact spaces generated by a space of cardinality bigger than $\mathfrak{m}_1$?

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Dieudonné complete spaces

Definition

A space X is said to be Dieudonné complete if there is a complete uniformity inducing its topology (i.e., the fine uniformity of X is complete).

Theorem (MH)

If X is a Dieudonné complete space and $\mathfrak{m}$ a measurable cardinal then the following properties are equivalent:

1. *X is κ-compact and $\mathfrak{m}(\kappa) = \mathfrak{m}, \kappa$ not measurable.*
2. *X is λ-ultracompact and $\mathfrak{m}(\lambda) = \mathfrak{m}, \lambda$ not measurable.*
3. *X contains no closed discrete subspace of cardinality $\mathfrak{m}$.*
4. *X is $H(\mu)$-compact for any μ with $\mathfrak{m}(\mu) = \mathfrak{m}$.*

Corollary

The class of Dieudonné spaces is simple iff the class of measurable

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For an infinite cardinal κ we denote by $H(\kappa)$ the metrizable hedgehog with κ many spines.

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Sequential continuity

Theorem (N.Varopoulos)

Every sequentially continuous homomorphism between compact groups of cardinalities less than $\mathfrak{m}_1$ *is continuous.*

Theorem (CH)

Every sequentially continuous mapping between compact groups of cardinalities less than $\mathfrak{m}_1$ *is continuous.*

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Factorizations of maps on products

Let $X \subset \prod_I X_i$ and $f : X \to Y$. We say that f depends on $J \subset I$ (or on $|J|$ coordinates, or that f factorizes via $\mathrm{pr}_J(X)$) if there exists a map $f_J : \mathrm{pr}_J(X) \to Y$ such that $f = f_J \circ \mathrm{pr}_J$, i.e., if $f(x) = f(y)$ provided $x, y \in X, \mathrm{pr}_J(x) = \mathrm{pr}_J(y)$.

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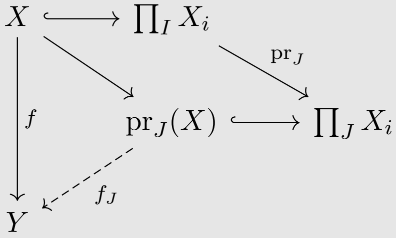

Σ-products

Σ-product

Let $p \in \prod_I X_i$. The subset $\{x \in \prod_I X_i; |i \in I; \mathrm{pr}_i a \neq \mathrm{pr}_i(p)| \leq \omega\}$ is called a Σ-product of $\{X_i\}_I$ with the basic point p.
If instead of $\leq \omega$ in the previous definition we use $< \omega$ we get σ-products.

Theorem (N.Noble)

If all $X_i, i \in I$, are first countable then every Σ-product of $\{X_i\}_I$ is a Fréchet space.

Corollary

Every sequentially continuous mapping on a product of first countable spaces is continuous on every Σ-product of $\{X_i\}_I$.

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Productivity number

Question

When a sequentially continuous map defined on a product of spaces is continuous?

Spaces having the property that every sequentially continuous map defined on them and ranging in a given class of spaces is continuous, form a coreflective class, i.e., the class is closed under taking quotients and inductive limits (sums). So, there is a more general question how big are the so called productivity numbers of coreflective classes:

Definition

Productivity number of a coreflective class $\mathcal{C}$ is the smallest cardinal κ such that a product $\prod_\kappa X_\alpha, X_\alpha \in \mathcal{C}$, does not belong to $\mathcal{C}$.

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Theorem (I.Glicksberg)

For infinite spaces X, Y the equality $\beta(X \times Y) = \beta(X) \times \beta(Y)$ holds iff $X \times Y$ is pseudocompact.

Theorem

Let X, Y have Ulam measurable cardinalities. If $\upsilon(X \times Y) = \upsilon(X) \times \upsilon(Y)$ then $X \times Y$ is pseudo-$\mathfrak{m}_1$-compact. The converse is not true.

Theorem

The property $\upsilon(X \times Y) = \upsilon(X) \times \upsilon(Y)$ for $X \times Y$ is not topological for infinite spaces X, Y.

Theorem

Let $\mathcal{K}$ be a finitely productive class of spaces containing all compact spaces and a pair P, Q with $\upsilon(P \times Q) \neq \upsilon(P) \times \upsilon(Q)$. Then there are no topological properties $\mathcal{A}, \mathcal{B}$ such that for $X, Y \in \mathcal{K}$ one has $\upsilon(X \times Y) = \upsilon(X) \times \upsilon(Y)$ iff $X, Y \in \mathcal{A}, X \times Y \in \mathcal{B}$.

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What is the situation for Hewitt-Nachbin realcompactificatrion υ? When $\upsilon(X \times Y) = \upsilon(X) \times \upsilon(Y)$? There is a partial analogous assertion to Glicksberg result:

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